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Pseudospectral Fourier reconstruction with the modified Inverse Polynomial Reconstruction Method

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ABSTRACT

We generalize the Inverse Polynomial Reconstruction Method (IPRM) for mitigation of the Gibbs phenomenon by reconstructing a function from its m lowest Fourier coefficients as an algebraic polynomial of degree at most $n-1$ $(m \geqslant n)$. We compute approximate Legendre coefficients of the function by solving a linear least squares problem. We show that if $m \geqslant n^2$, the condition number of the problem does not exceed 2.39. Consequently, if $m \geqslant n^2$, the convergence rate of the modified IPRM for an analytic function is root exponential on the whole interval of definition. Numerical stability and accuracy of the proposed algorithm are validated experimentally.

1. Introduction

The classical representation of a smooth and periodic function by its Fourier series is efficient and easy to use, and thus gives rise to a large class of numerical algorithms with spectral convergence, see [1]. By contrast, the representation of a non-periodic or discontinuous function by its Fourier series is less versatile because of the Gibbs phenomenon. Specifically, the partial sums of the Fourier series do not converge at a jump discontinuity, but merely oscillate within a certain range. Therefore a straightforward representation of a function with discontinuities via its Fourier series results in a relatively slow convergence rate. This behavior also occurs for analytic, but non-periodic functions.

Algorithms based on manipulation of the Fourier coefficients and techniques to combat the Gibbs phenomenon are still actively studied, as is evidenced by several recent contributions [2–5].

Historically the first methods for mitigation of the Gibbs phenomenon were based on projection and filtering, see a survey article [5]. A probabilistic approach was proposed in [6]. Recently, a substantial progress has been achieved by means of inverse methods, see for example [7]. In this paper, we focus on the Inverse Polynomial Reconstruction Method (IPRM) and its modification, and carry out a theoretical and experimental study of their properties.

The IPRM was introduced by Jung and Shizgal in order to remedy the Gibbs phenomenon, [8–11]. Their idea is to reconstruct a function f defined on a finite interval from n Fourier coefficients as an algebraic polynomial of degree at most $n-1$. Specifically, given the Fourier coefficients $\hat{f}(k),\ k = -\lfloor \frac{n-1}{2} \rfloor, \ldots, \lfloor \frac{n}{2} \rfloor$, the function f is approximated by a polynomial p of degree at most $n-1$ with identical Fourier coefficients

$$\hat{p}(k) = \hat{f}(k), \quad k = -\left\lfloor \frac{n-1}{2} \right\rfloor, \ldots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{1}$$

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They expand p with respect to the Gegenbauer polynomials (see [12, Sec. 8.93], for a definition)

$$p(x)=\sum_{l=0}^{n-1}a_l C_l^{\lambda}(x), \tag{2}$$

and determine the coefficients $a_0,\ldots,a_{n-1}$ from the following system of linear equations

$$\sum_{l=0}^{n-1}a_l\widehat{C_l^{\lambda}}(k)=\hat{f}(k),\quad k=-\left\lfloor\frac{n-1}{2}\right\rfloor,\ldots,\left\lfloor\frac{n}{2}\right\rfloor. \tag{3}$$

The effectiveness of this approach derives from the fact that a smooth function is efficiently approximated by its Gegenbauer series. In this framework, one can also use other polynomial bases depending on the problem at hand.

While a rigorous proof of the existence of the inverse polynomial reconstruction was published recently in [13], other basic aspects of the IPRM are still not fully understood. For example, numerical experiments reported in [11] indicate that the condition number of the linear system (3) grows exponentially with the dimension n. However, no formal proof of this observation is known.

Such extreme ill-conditioning leads to significant limitations of the IPRM. For example, the IPRM fails to converge if f is a meromorphic function with singularities located too close to its domain, see Section 6. Divergence occurs because the Gegenbauer series of f does not converge fast enough to compensate for the exponential growth of the condition numbers.

The most stable version of the IPRM presented in [11] uses a regularization procedure, which in some cases significantly improves accuracy of reconstruction. The regularization procedure employed there is the Gaussian elimination with truncation of all sufficiently small entries in the solution of the lower triangular stage.

The IPRM as formulated in [8–11] solves the typically ill-conditioned square linear system (3). We propose a modified version of the IPRM that reconstructs the polynomial p using more Fourier coefficients than the minimum required by its degree, and can be formulated as an overdetermined linear problem. We approximate the function f by a polynomial p of degree at most $n-1$ using information contained in the m lowest Fourier coefficients of f, with $m\geqslant n$. More precisely, the modified IPRM constructs a polynomial p such that the ℓ^2-norm of the difference of the m lowest Fourier coefficients

$$\left(\sum_{-\lfloor\frac{m-1}{2}\rfloor\leqslant k\leqslant\lfloor\frac{m}{2}\rfloor}|\hat{f}(k)-\hat{p}(k)|^2\right)^{\frac{1}{2}} \tag{4}$$

is minimal.

A matrix formulation of this problem can be obtained by expanding p with respect to the normalized Legendre polynomials (see Section 2 for a definition)

$$p(x)=\sum_{l=0}^{n-1}a_l\widetilde{P}_l(x). \tag{5}$$

With this representation of p, minimizing the expression in (4) amounts to the computation the coefficients $a_0,\ldots,a_{n-1}$ as the least squares solution of the overdetermined system of equations

$$\sum_{l=0}^{n-1}a_l\widehat{\widetilde{P}_l}(k)=\hat{f}(k)\quad k=-\left\lfloor\frac{m-1}{2}\right\rfloor,\ldots,\left\lfloor\frac{m}{2}\right\rfloor. \tag{6}$$

We show that the modified IPRM with $m\geqslant n^2$, dramatically improves the numerical conditioning of the method, and widens the scope of applications. Specifically, we derive three types of results and support them with numerical simulations.

1. *Solvability*. We show that a piecewise polynomial function with uniformly bounded degrees can be reconstructed from its consecutive Fourier coefficients, provided that the total number of the Fourier coefficients is greater than or equal to the number of unknown polynomial coefficients. We assume that the locations of discontinuities are known.
2. *Conditioning.* If $m\geqslant n^2$, then the condition number of the underlying least squares problem (6) does not exceed 2.39.
3. *Root-exponential convergence.* We demonstrate that for an analytic function f the resulting algorithm has a root-exponential convergence rate in terms of the number of Fourier coefficients, even if the analytic continuation of f has singularities near the real line. If p_m denotes the solution of (4) of degree at most $\sqrt{m}-1$ computed from m Fourier coefficients, then

$$\|f-p_m\|_{\infty}\leqslant c_1e^{-c_2\sqrt{m}}, \tag{7}$$

where the constants $c_1,c_2>0$ depend only the function f, see Section 5.2 for details.

The modified IPRM has important advantages in practical applications. First, the modified IPRM allows us to incorporate all available Fourier coefficients for reconstruction. Second, thanks to low condition numbers, we can reduce, and practically avoid, the amplification of any noise that is typically present in the measurements. Third, the method tells us how many Fourier coefficients suffice for an accurate representation of a smooth, but non-periodic function. For smooth and periodic functions, it is routine to truncate the Fourier series when the coefficients have decreased below a given threshold. A similar procedure is now available for non-periodic functions via the modified IPRM, even though their Fourier coefficients decay slowly.

Following a discussion in [14], we illustrate stability and accuracy of the modified IPRM as applied to the solution of a boundary value problem for the Poisson equation in one dimension. The resulting convergence rate is again root-exponential.

Numerical experiments indicate that $\mathcal{O}(n^2)$ Fourier coefficients are necessary for the stable reconstruction of a polynomial of degree $n-1$. We conjecture that using only $\mathcal{O}(n^\alpha)$ Fourier coefficients with $\alpha < 2$ leads to least squares problems with arbitrarily large condition numbers.

The paper is organized as follows. In Section 2 we review the definitions and properties of the Legendre polynomials and the spherical Bessel functions, and also some pertinent facts from linear algebra. In Section 3 we prove the existence of the IPRM reconstruction. Section 4 deals with stability of the IPRM and contains explicit estimates for the condition number of problem (6). We describe the algorithm of the modified IPRM in Section 5. We present the simulation results in Section 6. Our conclusions are given in Section 7.

2. Preliminaries

Our solution of the least squares problem (4) requires explicit expressions for the Fourier coefficients of the Legendre polynomials and some classical estimates for the spherical Bessel functions, which we present in this section for readers' convenience.

The spherical Bessel function of the first kind and order n, $n = 0, 1, \ldots$, is an entire function given by the following power series (see [15, Sec. 10.1.2])

$$j_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!(2m+2n+1)!!} x^{2m+n}, \tag{8}$$

where $(2k+1)!! = 1 \cdot 3 \cdot \ldots \cdot (2k+1)$. Clearly, j_n is an even function for even n, and an odd function for odd n. Explicit expressions for j_n are well-known, for example

$$j_0(x) = \frac{\sin x}{x}, \tag{9}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \tag{10}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x. \tag{11}$$

The function j_n is related to the Bessel function $J_{n+\frac{1}{2}}$ of the first kind and order $n+\frac{1}{2}$ in the following way (see [15, Sec. 10.1.1])

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x). \tag{12}$$

For every $x > n + \frac{1}{2}$, we have the inequality (see [12, Sec. 8.479])

$$\frac{\pi}{2} J^2_{n+\frac{1}{2}}(x) < \frac{1}{\sqrt{x^2 - \left(n+\frac{1}{2}\right)^2}}. \tag{13}$$

Consequently, for every $x > n + \frac{1}{2}$

$$j_n^2(x) < \frac{1}{x\sqrt{x^2 - \left(n+\frac{1}{2}\right)^2}}. \tag{14}$$

The Legendre polynomial P_n, $n = 0, 1, \ldots$, is defined by the formula (see [15, Sec. 22.11.5])

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2-1)^n]. \tag{15}$$

The Legendre polynomials are orthogonal on the interval $[-1, 1]$, but not orthonormal. It is convenient to use the normalized Legendre polynomials (see [15, Sec. 22.2.10])

$$\widetilde{P}_n = \sqrt{n + \frac{1}{2}} P_n. \tag{16}$$

The Fourier coefficients of the Legendre polynomials can be expressed in terms of the spherical Bessel functions as (see [12, Sec. 7.243])

$$\widehat{P_n}(k) = \frac{1}{\sqrt{2}} \int_{-1}^{1} e^{-\imath k\pi x} P_n(x) dx = \sqrt{2}(-\imath)^n j_n(k\pi). \tag{17}$$

Consequently, the Fourier coefficients of the normalized Legendre polynomials are given by the following formula

$$\widehat{\widetilde{P}_n}(k) = (-\imath)^n \sqrt{2n+1} j_n(k\pi), \quad k \in \mathbb{Z}. \tag{18}$$

Eq. (18) may be derived from an addition formula describing the expansion of a plane wave in terms of the Legendre polynomials ([15, Sec. 10.1.47]),

$$e^{\imath xt} = \sum_{n=0}^{\infty} (2n+1)\imath^n j_n(t) P_n(x). \tag{19}$$

We note that the Legendre polynomials are a special case of the Gegenbauer polynomials, see [12, Sec. 8.93] for a definition. Formula (18) has an analogue for the Gegenbauer polynomials, which is used in analysis of another method of resolution of the Gibbs phenomenon presented in [5].

In our estimates of matrix eigenvalues, we use Gerschgorin's theorem for Hermitian matrices (see [16, Theorem 7.2.1]).

Theorem 2.1. *If $A = (a_{ij})$ is an $n \times n$ Hermitian matrix, and $r_i = \sum_{j \neq i} |a_{ij}|$, for $i = 1, \ldots, n$, then each eigenvalue of A is located in at least one of the intervals $[a_{ii} - r_i, a_{ii} + r_i]$, $i = 1, \ldots, n$.*

Finally, we recall basic facts about the solution of a linear least squares problem. If A is an $m \times n$ complex matrix with $m \geqslant n$, then the reduced singular value decomposition of A has the form

$$A = U\Sigma V^*, \tag{20}$$

where U is an $m \times n$ isometry matrix, Σ is a diagonal $n \times n$ matrix with non-negative entries, and V is an $n \times n$ unitary matrix. Throughout this paper, the conjugate transpose of a matrix X is denoted by X^*. The diagonal entries $\sigma_1 \geqslant \cdots \geqslant \sigma_n \geqslant 0$ of Σ are called the singular values of A. If A has full rank and $b \in \mathbb{C}^m$, then the (overdetermined) least squares problem

$$\min_{x \in \mathbb{C}^n} \|Ax - b\| \tag{21}$$

has a unique solution given by the formula (see [17, Theorem 1.2.10])

$$x = V\Sigma^{-1}U^*b. \tag{22}$$

Throughout this paper, the Moore–Penrose pseudoinverse of the matrix A is denoted by $A^\dagger$. If A has full rank, then

$$A^\dagger = V\Sigma^{-1}U^* \tag{23}$$

(see [17, Definition 1.2.1]). Clearly, $A^\dagger$ is an $n \times m$ matrix, and $A^\dagger A$ is the $n \times n$ identity matrix. Moreover, the operator norm of $A^\dagger$ is the reciprocal of the smallest singular value σ_n,

$$\|A^\dagger\| = \frac{1}{\sigma_n}. \tag{24}$$

Pseudoinverses are commonly used for regularization of ill-conditioned problems. In this paper, we use pseudoinverses of very well-conditioned, but rectangular matrices.

3. Existence of the IPRM reconstruction

In this section, we formulate sufficient conditions for a piecewise polynomial function to be uniquely determined by a finite number of its Fourier coefficients. As a consequence, we obtain an elementary proof of the existence of the IPRM reconstruction.

For a fixed sequence $\mathbf{a} = (a_0, \ldots, a_L)$ with the property

$$-1 = a_0 < a_1 < \cdots < a_{L-1} < a_L = 1, \tag{25}$$

we consider all piecewise polynomial functions defined on the interval $[-1, 1]$, with discontinuities possibly occurring only at the points $a_1, \ldots, a_{L-1}$. For every non-negative integer M, we denote by $\mathscr{P}_{M,\mathbf{a}}$ the linear space of all functions whose restrictions to each interval (a_{j-1}, a_j), $j = 1, \ldots, L$, are polynomials of degree not exceeding M. Since the sequence $\mathbf{a}$ defines a partition of $[-1, 1]$ into L subintervals, the dimension of $\mathscr{P}_{M,\mathbf{a}}$ equals $L(M+1)$.

Clearly, one needs at least $L(M+1)$ Fourier coefficients in order to uniquely reconstruct a function from $\mathscr{P}_{M,\mathbf{a}}$, but this requirement is far from sufficient. For instance, the sign (signum) function $\text{sgn}(x)$ defined on $(-1, 1)$ has all even Fourier coefficients equal to zero. Motivated by this example, we consider reconstruction from *consecutive* Fourier coefficients. The following result asserts the existence of the IPRM reconstruction for piecewise polynomial functions.

Theorem 3.1. *Let d and D be integers such that $d \leqslant 0 \leqslant D$, and let $p \in \mathscr{P}_{M,\mathbf{a}}$ have $D - d + 1$ consecutive vanishing Fourier coefficients*

$$\hat{p}(d) = \hat{p}(d+1) = \cdots = \hat{p}(D-1) = \hat{p}(D) = 0. \tag{26}$$

If $D-d+1 \geqslant L(M+1)$, then $p=0$ identically. Equivalently, every piecewise polynomial $p \in \mathscr{P}_{M,\mathbf{a}}$ is uniquely determined by its Fourier coefficients $\hat{p}(d), \ldots, \hat{p}(D)$.

The proof of the theorem is preceded by two lemmas. We omit a proof of the first lemma, which is a straightforward exercise in integration by parts. For every integer $m \geqslant 0,\ p^{(m)}$ denotes the mth derivative of p.

Lemma 3.1. *If p is a polynomial of degree at most M, and f is a function of class $C^{M+1}[a,b]$, then*

$$\int_a^b p(x) f^{(M+1)}(x)dx = \sum_{m=0}^{M}(-1)^m\left[p^{(m)}(b)f^{(M-m)}(b) - p^{(m)}(a)f^{(M-m)}(a)\right]. \tag{27}$$

For fixed integers d and D such that $d \leqslant D$, we denote by $\mathscr{T}_{d,D}$ the space of all *trigonometric* polynomials with period 2, whose spectrum is contained in the interval $[d\pi, D\pi]$, that is,

$$\mathscr{T}_{d,D} = \text{span}\{e^{\imath k\pi x} : d \leqslant k \leqslant D\}. \tag{28}$$

Lemma 3.2. *If $D-d+1 \geqslant L(M+1)$, then for all integers r and s such that $0 \leqslant r \leqslant L$ and $0 \leqslant s \leqslant M$, there exists a trigonometric polynomial $t \in \mathscr{T}_{d,D}$ that satisfies the following conditions*

1. $t^{(m)}(a_j) = 0$ *if* $j \neq r,\ 1 \leqslant j \leqslant L,\ 0 \leqslant m \leqslant M$,
2. $t^{(m)}(a_r) = 0$ *if* $m \leqslant s-1$,
3. $t^{(s)}(a_r) \neq 0$.

Proof. We define t by the formula

$$t(x) = e^{\imath d\pi x}(e^{\imath\pi x} - e^{\imath\pi a_r})^s \prod_{\substack{1\leqslant j\leqslant L \\ j \neq r}} (e^{\imath\pi x} - e^{\imath\pi a_j})^{M+1}. \tag{29}$$

In view of (25), the a_j's are distinct modulo 2. Therefore t has a zero of order exactly s at a_r, and a zero of order exactly $M+1$ at a_j for $j \neq r$, which implies conditions 1–3. The lowest frequency present in t is $d\pi$, and the highest frequency is $[(L-1)(M+1)+s+d]\pi$. Our assumptions imply that $(L-1)(M+1)+s+d \leqslant L(M+1)-1+d \leqslant D$, therefore $t \in \mathscr{T}_{d,D}$. □

Proof of Theorem 3.1. Since all the Fourier coefficients $\hat{p}(k),\ d \leqslant k \leqslant D$, are zero, for every trigonometric polynomial $t \in \mathscr{T}_{d,D}$ we have

$$\int_{-1}^{1} p(x)\overline{t(x)}dx = 0. \tag{30}$$

Moreover, its $(M+1)$st derivative $t^{(M+1)}$ is also in $\mathscr{T}_{d,D}$, which implies that

$$\int_{-1}^{1} p(x)\overline{t^{(M+1)}(x)}dx = 0. \tag{31}$$

Using Lemma 3.1 with $f = \bar{t}$, and denoting by p_j the restriction of p to the interval $(a_{j-1}, a_j),\ 1 \leqslant j \leqslant L$, Eq. (31) can be written as follows

$$\sum_{j=1}^{L}\sum_{m=0}^{M}(-1)^m\left[p_j^{(m)}(a_j)\overline{t^{(M-m)}(a_j)} - p_j^{(m)}(a_{j-1})\overline{t^{(M-m)}(a_{j-1})}\right] = 0. \tag{32}$$

Since t has period 2, the derivative $t^{(M-m)}$ has the same values at $a_0 = -1$ and $a_L = 1$, i.e. $t^{(M-m)}(a_0) = t^{(M-m)}(a_L)$. Therefore (32) is equivalent to the following

$$\sum_{m=0}^{M}\sum_{j=1}^{L-1}(-1)^m\left[p_j^{(m)}(a_j) - p_{j+1}^{(m)}(a_j)\right]\overline{t^{(M-m)}(a_j)} + \sum_{m=0}^{M}(-1)^m\left[p_L^{(m)}(a_L) - p_1^{(m)}(a_0)\right]\overline{t^{(M-m)}(a_L)} = 0. \tag{33}$$

With the aid of Lemma 3.2, we demonstrate that the coefficient of $\overline{t^{(M-m)}(a_j)}$ in Eq. (33) vanishes for every $j = 1, \ldots, L$ and $m = 0, \ldots, M$. Specifically, for each $r,\ 1 \leqslant r \leqslant L$, there exist trigonometric polynomials $t_0, \ldots, t_M \in \mathscr{T}_{d,D}$ such that

1. $t_s^{(m)}(a_j) = 0$ if $j \neq r,\ 1 \leqslant j \leqslant L,\ 0 \leqslant m \leqslant M$,
2. $t_s^{(m)}(a_r) = 0$ if $m \leqslant s-1$,
3. $t_s^{(s)}(a_r) \neq 0$.

Substituting the functions $t_0, \ldots, t_M$ into formula (33), we obtain a triangular system of $M+1$ linear equations with nonzero entries on the diagonal, whose solution are the coefficients of $\overline{t^{(M-m)}(a_r)}$ for $m = 0, \ldots, M$. Such a linear system has only the trivial solution. We have thus shown that

$$p_j^{(m)}(a_j) - p_{j+1}^{(m)}(a_j) = 0, \tag{34}$$
$$p_L^{(m)}(a_L) - p_1^{(m)}(a_0) = 0, \tag{35}$$

for $j = 1, \dots, L-1$ and $m = 0, \dots, M$. Since all the polynomials $p_j,\ 1 \leqslant j \leqslant L$, have degrees not exceeding M, (34) implies that the polynomials p_j are actually the restrictions of a single polynomial p to the intervals (a_{j-1}, a_j), respectively. Moreover, (35) implies that polynomial p is periodic with period $a_L - a_0 = 2$, and therefore constant. Finally, $\hat{p}(0) = 0$, so $p = 0$ identically. □

Let us consider a special case with $L = 1$. It follows from Theorem 3.1 that a polynomial p of degree at most $n-1$ is uniquely determined by its n lowest Fourier coefficients $\hat{p}(k),\ k = -\lfloor \frac{n-1}{2} \rfloor, \dots, \lfloor \frac{n}{2} \rfloor$. Expanding p into the normalized Legendre polynomials, we obtain the following corollary.

Corollary 3.1. *If $m \geqslant n$, then the matrix with the entries $a_{kl} = \widehat{\widetilde{P}_l}(k),\ k = -\lfloor \frac{m-1}{2} \rfloor, \dots, \lfloor \frac{m}{2} \rfloor,\ l = 0, \dots, n-1$, has full rank.*

4. Stability of the modified reconstruction

In this section, we discuss the gains in numerical stability that are achieved by the modified IPRM. Specifically, we show that the computation of the first n Legendre coefficients from the lowest αn^2 Fourier coefficients has condition number arbitrarily close to 1, if α is taken large enough. The results of this section are essential for our proof that the modified IPRM has a root-exponential rate of convergence.

For $n = 1, 2, \dots$, let A_n be the infinite matrix of the Fourier coefficients of the normalized Legendre polynomials $\widetilde{P}_l,\ l = 0, \dots, n-1$, defined by Eq. (16). In view of (18), the entries of A_n are given by the formula

$$a_{kl} = \widehat{\widetilde{P}_l}(k) = (-\imath)^l \sqrt{2l+1}\, j_l(k\pi), \tag{36}$$

for $k \in \mathbb{Z}$, $l = 0, \dots, n-1$. According to Plancherel's theorem, the columns of A_n are orthonormal in $\ell^2(\mathbb{Z})$, and the operator $A_n : \mathbb{C}^n \to \ell^2$ is an isometry. For fixed positive integers m and n, we denote by $A_{m,n}$ the finite $m \times n$ submatrix of A_n corresponding to the indices $k = -\lfloor \frac{m-1}{2} \rfloor, \dots, \lfloor \frac{m}{2} \rfloor$ and $l = 0, \dots, n-1$. We expect that if m is sufficiently large with respect to n, then the condition number of the matrix $A_{m,n}$ is arbitrarily close to 1. The following three theorems formulate quantitative versions of this observation.

Theorem 4.1. *For every $n = 1, 2, \dots$, and every integer $m \geqslant 2n$, the smallest singular value $\sigma_{m,n}$ of the matrix $A_{m,n}$ satisfies the following inequality*

$$\sigma_{m,n}^2 \geqslant 1 - \frac{4}{\pi} n \arcsin\left(\frac{2}{\pi}\frac{n}{m}\right). \tag{37}$$

Proof. Let $B_{m,n} = A_{m,n}^* A_{m,n}$. Clearly, the eigenvalues of the $n \times n$ matrix $B_{m,n}$ are the squares of the singular values of the $m \times n$ matrix $A_{m,n}$. Our approach is to use Gerschgorin's circle theorem (Theorem 2.1) to estimate the eigenvalues of $B_{m,n}$ from below.

The entries $b_{pq},\ 0 \leqslant p, q \leqslant n-1$, of $B_{m,n}$ are given by the formula

$$b_{pq} = \sum_{-\lfloor \frac{m-1}{2} \rfloor \leqslant k \leqslant \lfloor \frac{m}{2} \rfloor} \overline{a_{kp}} a_{kq}. \tag{38}$$

If $p \neq q$, then the sequences of the Fourier coefficients of $\widetilde{P}_p$ and $\widetilde{P}_q$ are orthogonal in $\ell^2(\mathbb{Z})$, and we have

$$\sum_{k \in \mathbb{Z}} \overline{a_{kp}} a_{kq} = 0. \tag{39}$$

We can express b_{pq} as the sum over the complementary set of indices k. Specifically, if $I_m = \{k \in \mathbb{Z} : k < -\lfloor \frac{m-1}{2} \rfloor \text{ or } k > \lfloor \frac{m}{2} \rfloor\}$, then

$$b_{pq} = -\sum_{k \in I_m} \overline{a_{kp}} a_{kq}. \tag{40}$$

Consequently, we have the following estimate for the pth Gerschgorin radius r_p

$$r_p = \sum_{q \neq p} |b_{pq}| \leqslant \sum_{q \neq p} \sum_{k \in I_m} |a_{kp}||a_{kq}|. \tag{41}$$

For $p = q$, since the Fourier coefficients of $\widetilde{P}_p$ are normalized in $\ell^2(\mathbb{Z})$, we obtain

$$b_{pp} = \sum_{-\lfloor \frac{m-1}{2} \rfloor \leqslant k \leqslant \lfloor \frac{m}{2} \rfloor} \overline{a_{kp}} a_{kp} = 1 - \sum_{k \in I_m} |a_{kp}|^2. \tag{42}$$

Combining (41) and (42), we obtain

$$b_{pp} - r_p \geqslant 1 - \sum_{q=0}^{n-1} \sum_{k \in I_m} |a_{kp}||a_{kq}| = 1 - \sum_{k \in I_m} \sum_{q=0}^{n-1} |a_{kp}||a_{kq}|. \tag{43}$$

Using (14) and (36), the entries of $A_{m,n}$ are estimated as follows

$$|a_{kq}| = \sqrt{2q+1}|j_q(k\pi)| = \sqrt{2q+1}|j_q(|k|\pi)| \leqslant \frac{\sqrt{2q+1}}{\sqrt{|k|\pi}\sqrt[4]{k^2\pi^2 - (q+\frac{1}{2})^2}} \leqslant \frac{\sqrt{2n-1}}{\sqrt{|k|\pi}\sqrt[4]{k^2\pi^2 - (n-\frac{1}{2})^2}}. \tag{44}$$

Substituting (44) into (43), we obtain

$$b_{pp} - r_p \geqslant 1 - \sum_{k \in I_m} \frac{n(2n-1)}{|k|\pi\sqrt{k^2\pi^2 - (n-\frac{1}{2})^2}}. \tag{45}$$

Since $\lfloor \frac{m}{2} \rfloor + 1 \geqslant \lfloor \frac{m-1}{2} \rfloor + 1 \geqslant \frac{m}{2}$, the sum in (45) can be estimated as follows

$$\begin{aligned} \sum_{k \in I_m} \frac{n(2n-1)}{|k|\pi\sqrt{k^2\pi^2 - (n-\frac{1}{2})^2}} &= \sum_{k \geqslant \lfloor \frac{m}{2} \rfloor + 1} \frac{n(2n-1)}{k\pi\sqrt{k^2\pi^2 - (n-\frac{1}{2})^2}} + \sum_{k \geqslant \lfloor \frac{m-1}{2} \rfloor + 1} \frac{n(2n-1)}{k\pi\sqrt{k^2\pi^2 - (n-\frac{1}{2})^2}} \\ &\leqslant 2\sum_{k \geqslant \frac{m}{2}} \frac{n(2n-1)}{k\pi\sqrt{k^2\pi^2 - (n-\frac{1}{2})^2}}. \end{aligned} \tag{46}$$

We now estimate the last sum in (46) using Lemma 8.1 (proved in Appendix A) with $M = \frac{m}{2}$ and $u = \frac{1}{\pi}(n - \frac{1}{2})$

$$\sum_{k \in I_m} \frac{n(2n-1)}{k\pi\sqrt{k^2\pi^2 - (n-\frac{1}{2})^2}} \leqslant 2\sum_{k \geqslant \frac{m}{2}} \frac{n(2n-1)}{k\pi\sqrt{k^2\pi^2 - (n-\frac{1}{2})^2}} \leqslant \frac{4}{\pi} n \arcsin\left(\frac{2}{\pi}\frac{n-\frac{1}{2}}{m-1}\right). \tag{47}$$

Substituting (47) into (45), and using the assumption that $m \geqslant 2n$, we obtain

$$b_{pp} - r_p \geqslant 1 - \frac{4}{\pi} n \arcsin\left(\frac{2}{\pi}\frac{n-\frac{1}{2}}{m-1}\right) \geqslant 1 - \frac{4}{\pi} n \arcsin\left(\frac{2}{\pi}\frac{n}{m}\right). \tag{48}$$

It follows from Gerschgorin's theorem that all the eigenvalues of the matrix $B_{m,n}$ are greater than or equal to $1 - \frac{4}{\pi} n \arcsin(\frac{2}{\pi}\frac{n}{m})$. □

Theorem 4.2. *For every $\alpha \geqslant 1$, every $n = 1, 2, \ldots$, and every integer $m \geqslant \alpha n^2$, the smallest singular value $\sigma_{m,n}$ of the matrix $A_{m,n}$ satisfies the following inequality*

$$\sigma_{m,n}^2 \geqslant 1 - \frac{8}{\pi} \arcsin \frac{1}{\pi\alpha}. \tag{49}$$

Proof. Inequality (49) holds for $n = 1$, because

$$\sigma_{m,1}^2 = \sum_{-\lfloor \frac{m-1}{2} \rfloor \leqslant k \leqslant \lfloor \frac{m}{2} \rfloor} j_0^2(k\pi) = 1, \tag{50}$$

where $j_0(x) = \frac{\sin x}{x}$ according to (9). Let us then assume that $n \geqslant 2$, and therefore $m \geqslant \alpha n^2 \geqslant 2n$. We can now use Theorem 4.1 to conclude that

$$\sigma_{m,n}^2 \geqslant 1 - \frac{4}{\pi} n \arcsin\left(\frac{2}{\pi}\frac{n}{m}\right) \geqslant 1 - \frac{4}{\pi} n \arcsin \frac{2}{\pi\alpha n}. \tag{51}$$

The substitution $x = \frac{2}{\pi\alpha n}$ reduces the last term in (51) to a scalar multiple of the function $f(x) = \frac{\arcsin x}{x}$. The claim now follows from the fact that the function f is increasing on the interval $(0, 1)$. □

Theorem 4.3. *For every $\alpha \geqslant 1$, every $n = 1, 2, \ldots$, and every integer $m \geqslant \alpha n^2$, the condition number $\kappa(A_{m,n})$ of the matrix $A_{m,n}$ satisfies the following inequality*

$$\kappa(A_{m,n}) \leqslant \left(1 - \frac{8}{\pi} \arcsin \frac{1}{\pi\alpha}\right)^{-\frac{1}{2}}. \tag{52}$$

Proof. Since $A_{m,n}$ is the restriction of the isometry A_n to m dimensions, $A_{m,n}$ is a contraction, and its singular values do not exceed 1. Therefore the condition number of $A_{m,n}$ does not exceed $\frac{1}{\sigma_{m,n}}$, and (52) follows from Theorem 4.2. □

For $\alpha = 1$, the right hand side of inequality (52) is less than 2.39, while an experimentally computed maximum of $\kappa(A_{n^2,n})$ equals ≈1.21, see Section 6.1. Moreover, our numerical simulations indicate that for any fixed value $\alpha > 0$, the condition number of $A_{\lceil \alpha n^2 \rceil, n}$ remains uniformly bounded with respect to n.

Using more complex arguments, one can also obtain an upper bound on the condition number of reconstruction with the Legendre polynomials in the more general case of a piecewise polynomial function in the space $\mathscr{P}_{M,\mathbf{a}}$ defined in the previous section.

5. Modified reconstruction algorithm

We now present the modified version of the IPRM algorithm and analyze its properties. We demonstrate that for analytic functions the algorithm has a root-exponential convergence rate on the whole interval of definition.

5.1. Description of the algorithm

The modified IPRM algorithm finds a truncated Legendre series of a given function from its truncated Fourier series by solving a rectangular least squares problem.

We fix positive integers m and n such that $m \geqslant n$. We assume that we are given the m lowest Fourier coefficients of an unknown function f defined of the interval $[-1,1]$. More precisely, the coefficients $\hat{f}(d), \ldots, \hat{f}(D)$ are known, where $d = -\lfloor \frac{m-1}{2} \rfloor$ and $D = \lfloor \frac{m}{2} \rfloor$. We approximate the function f by an algebraic polynomial p of degree at most $n-1$ such that the ℓ^2-norm of the difference of the Fourier coefficients

$$\left(\sum_{d \leqslant k \leqslant D} |\hat{p}(k) - \hat{f}(k)|^2 \right)^{\frac{1}{2}} \tag{53}$$

is minimal. In order to derive a matrix formulation, we denote by $\mathbf{c} = [c_0, \ldots, c_{n-1}]^T$ the normalized Legendre coefficients of p, so that

$$p(x) = \sum_{l=0}^{n-1} c_l \widetilde{P}_l(x). \tag{54}$$

In Section 4, we have introduced the $m \times n$ matrix $A_{m,n}$, whose entries a_{kl} are the Fourier coefficients of the normalized Legendre polynomials

$$a_{kl} = \widehat{\widetilde{P}_l}(k) = (-\imath)^l \sqrt{2l+1} j_l(k\pi), \tag{55}$$

$k = d, \ldots, D,\ l = 0, \ldots, n-1$. Consequently,

$$[\hat{p}(d), \ldots, \hat{p}(D)]^T = A_{m,n}\mathbf{c}, \tag{56}$$

and

$$\left(\sum_{d \leqslant k \leqslant D} |\hat{p}(k) - \hat{f}(k)|^2 \right)^{\frac{1}{2}} = \|A_{m,n}\mathbf{c} - [\hat{f}(d), \ldots, \hat{f}(D)]^T\|. \tag{57}$$

In order to construct an approximation of f from its Fourier coefficients, we use the following reconstruction algorithm.

1. Solve the overdetermined least squares problem for approximate Legendre coefficients $\mathbf{c} = [c_0, \ldots, c_{n-1}]^T$

$$\min_{\mathbf{c} \in \mathbb{C}^n} \|A_{m,n}\mathbf{c} - [\hat{f}(d), \ldots, \hat{f}(D)]^T\|. \tag{58}$$

Specifically, we use the pseudoinverse solution

$$\mathbf{c} = A^{\dagger}_{m,n}[\hat{f}(d), \ldots, \hat{f}(D)]^T, \tag{59}$$

described in Section 2.

2. Approximate f by a truncated Legendre series using the estimated Legendre coefficients $\mathbf{c}$

$$f_{m,n} = \sum_{l=0}^{n-1} c_l \widetilde{P}_l. \tag{60}$$

The matrix $A_{m,n}$ is of full rank, as stated in Corollary 3.1. Although we state the algorithm for general parameters m and n ($m \geqslant n$), its intended use is with a tight control over the condition numbers. Thus in a typical scenario, one assumes that $m \geqslant n^2$, as stated in Theorem 4.3.

5.2. Rate of convergence

The rate of convergence of this method depends on the parameters m and n, and on the smoothness of the function. It is well-known that an analytic function possesses an efficient representation by its Legendre series. The rate of convergence of the Legendre series is determined by the quasi-radial distance from the interval $[-1,1]$ to the nearest singularity measured in the elliptic coordinate system [1]. Specifically, let $\mu > 0$, and let f be a function analytic inside the ellipse

$$\frac{x^2}{\cosh^2\mu} + \frac{y^2}{\sinh^2\mu} = 1. \tag{61}$$

The ellipse has its foci at $z = \pm 1$, in particular f is defined on the interval $[-1,1]$. Under these assumptions, the Legendre series of $f = \sum_{l=0}^{\infty} a_l \widetilde{P}_l$ converges exponentially, and the Legendre coefficients a_l satisfy the inequality

$$\limsup_{l\to\infty} |a_l|^{\frac{1}{l}} \leqslant e^{-\mu}. \tag{62}$$

Equivalently, for every β smaller than μ, there is a constant $c = c(\beta)$ such that

$$|a_l| \leqslant ce^{-\beta l}, \quad l = 0, 1, \ldots. \tag{63}$$

For such functions, the convergence rate of the modified IPRM is exponential in terms of the number of Legendre coefficients.

Theorem 5.1. *Let us assume that* $f = \sum_{l=0}^{\infty} a_l \widetilde{P}_l$ *has exponentially decreasing Legendre coefficients, i.e.*

$$|a_l| \leqslant ce^{-\beta l}, \quad l = 0, 1, \ldots, \tag{64}$$

for fixed numbers $c > 0$ *and* $\beta > 0$. *If* $f_{m,n}$ *defined by Eq. (60) is the reconstruction obtained via the modified IPRM, then there exists* $\gamma = \gamma(\beta) > 0$ *such that for every* $n = 1, 2, \ldots$ *and every integer* $m \geqslant n^2$,

$$\|f - f_{m,n}\|_\infty \leqslant c\gamma n e^{-\beta n}. \tag{65}$$

Proof. By construction,

$$f_{m,n} = \sum_{l=0}^{n-1} c_l \widetilde{P}_l, \tag{66}$$

where

$$\mathbf{c} = A^{\dagger}_{m,n} [\hat{f}(d), \ldots, \hat{f}(D)]^T, \tag{67}$$

and $d = -\lfloor \frac{m-1}{2} \rfloor$ and $D = \lfloor \frac{m}{2} \rfloor$. To obtain an error estimate, we use the triangle inequality, and then treat the two resulting terms separately

$$\|f - f_{m,n}\|_\infty \leqslant \left\| f - \sum_{l=0}^{n-1} a_l \widetilde{P}_l \right\|_\infty + \left\| \sum_{l=0}^{n-1} (a_l - c_l) \widetilde{P}_l \right\|_\infty \tag{68}$$

$$= \left\| \sum_{l=n}^{\infty} a_l \widetilde{P}_l \right\|_\infty + \left\| \sum_{l=0}^{n-1} (a_l - c_l) \widetilde{P}_l \right\|_\infty. \tag{69}$$

Since $\|\widetilde{P}_l\|_\infty = \sqrt{l + \frac{1}{2}}$, the first sum is bounded as follows

$$\left\| \sum_{l=n}^{\infty} a_l \widetilde{P}_l \right\|_\infty \leqslant \sum_{l=n}^{\infty} |a_l| \|\widetilde{P}_l\|_\infty \leqslant \sum_{l=n}^{\infty} ce^{-\beta l} \sqrt{l + \frac{1}{2}} \leqslant c\gamma_1 \sqrt{n} e^{-\beta n}, \tag{70}$$

where $\gamma_1 = \gamma_1(\beta) > 0$. To deal with the second sum, we use the Cauchy–Schwarz inequality

$$\left\| \sum_{l=0}^{n-1} (a_l - c_l) \widetilde{P}_l \right\|_\infty \leqslant \sum_{l=0}^{n-1} |a_l - c_l| \sqrt{l + \frac{1}{2}} \leqslant \|\mathbf{a} - \mathbf{c}\| \frac{n}{\sqrt{2}}, \tag{71}$$

where $\mathbf{a} = [a_0, \ldots, a_{n-1}]^T$. Setting $L_n = \sum_{l=0}^{n-1} a_l \widetilde{P}_l$, we have

$$A^{\dagger}_{m,n} [\widehat{L_n}(d), \ldots, \widehat{L_n}(D)]^T = A^{\dagger}_{m,n} A_{m,n} \mathbf{a} = \mathbf{a}. \tag{72}$$

Using (64) and (67), we obtain

$$\|\mathbf{a} - \mathbf{c}\| \leqslant \|A^{\dagger}_{m,n}\| \left(\sum_{k=d}^{D} |\widehat{L_n}(k) - \hat{f}(k)|^2 \right)^{\frac{1}{2}} \leqslant \|A^{\dagger}_{m,n}\| \left\| \sum_{l=0}^{n-1} a_l \widetilde{P}_l - f \right\|_{L^2} = \|A^{\dagger}_{m,n}\| \left(\sum_{l=n}^{\infty} |a_l|^2 \right)^{\frac{1}{2}} \leqslant \|A^{\dagger}_{m,n}\| c (1 - e^{-2\beta})^{-\frac{1}{2}} e^{-\beta n}. \tag{73}$$

According to formula (24) and Theorem 4.2, $\|A^{\dagger}_{m,n}\| \leqslant \left(1-\frac{8}{\pi}\arcsin\frac{1}{\pi}\right)^{-\frac{1}{2}} < 2.39$. Combining this with Eqs. (71) and (73), we obtain our final estimate for the second term

$$\left\|\sum_{l=0}^{n-1}(a_l-c_l)\widetilde{P}_l\right\|_{\infty} \leqslant c\gamma_2 n e^{-\beta n}, \tag{74}$$

where $\gamma_2=\gamma_2(\beta)>0$. □

Estimate (65) implies that the convergence rate is root-exponential in terms of the number of Fourier coefficients m, if m is kept proportional to n^2. For example, if $n=\lfloor\sqrt{m}\rfloor$, then

$$\|f-f_{m,n}\|_{\infty} \leqslant c\gamma e^{\beta}\sqrt{m}e^{-\beta\sqrt{m}} \leqslant \frac{2c\gamma}{\beta}e^{\beta}e^{-\frac{\beta}{2}\sqrt{m}}. \tag{75}$$

Inequality (7) in Section 1 is thus demonstrated.

5.3. Complexity

In Step 1, we compute the approximate Legendre coefficients $\mathbf{c}$ by applying the pseudoinverse $A^{\dagger}_{m,n}$ to m Fourier coefficients of f. The standard way to compute the pseudoinverse uses the singular value decomposition via formula (23), and has a complexity of $\mathcal{O}(mn^2)$. Fortunately, it can be reduced to $\mathcal{O}(mn)$ by solving the least squares problem (58) iteratively with the conjugate gradient method applied to the normal equations. This is accomplished, for example, with the well-tested LSQR algorithm, see [18]. In the setting of Theorem 4.3, the condition number of $A_{m,n}$ is close to 1.0, which dramatically accelerates convergence of the conjugate gradient method. It is well-known that the conjugate gradient method applied to a matrix with condition number κ converges exponentially at the rate of $\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$, see Theorem 10.2.6 in [16]. Therefore, the least squares problem with dimensions $m\times n$ can be solved to a relative precision ε in $O(mn\log\varepsilon)$ operations. For reasons of stability, we typically choose $n=\mathcal{O}(\sqrt{m})$, which results in a complexity of $O(m^{1.5}\log\varepsilon)$. In our experiments, double precision is reached in five iterations of LSQR, see Fig. 4.

In Step 2, we compute the normalized Legendre polynomials by the standard three-term recursion, and then find the sum in (60). The evaluation of $f_{m,n}$ at N nodes in $[-1,1]$ requires additional $\mathcal{O}(nN)$ flops.

6. Simulation results

6.1. Condition numbers

Throughout this section, we use the $m\times n$ matrix $A_{m,n}$ of the Fourier coefficients of the Legendre polynomials with the entries

$$a_{kl}=\widehat{\widetilde{P}_l}(k)=(-\imath)^l\sqrt{2l+1}j_l(k\pi), \tag{76}$$

$k=-\lfloor\frac{m-1}{2}\rfloor,\ldots,\lfloor\frac{m}{2}\rfloor,\ l=0,\ldots,n-1.$

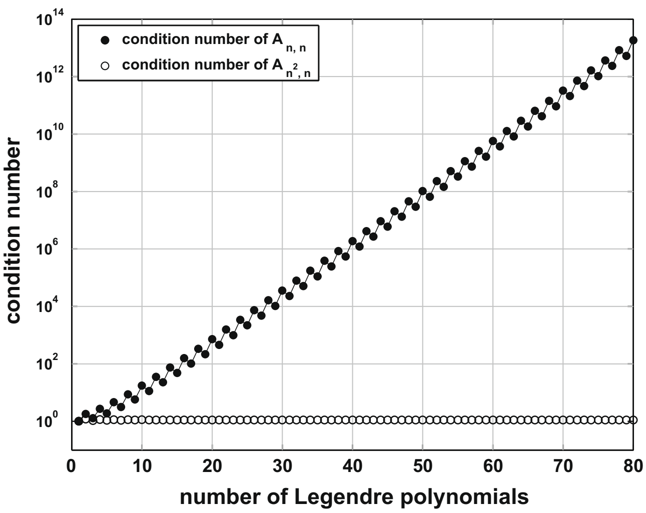

Fig. 1. The condition numbers of the matrices $A_{n,n}$ and $A_{n^2,n}$ for $n=1,\ldots,80$.

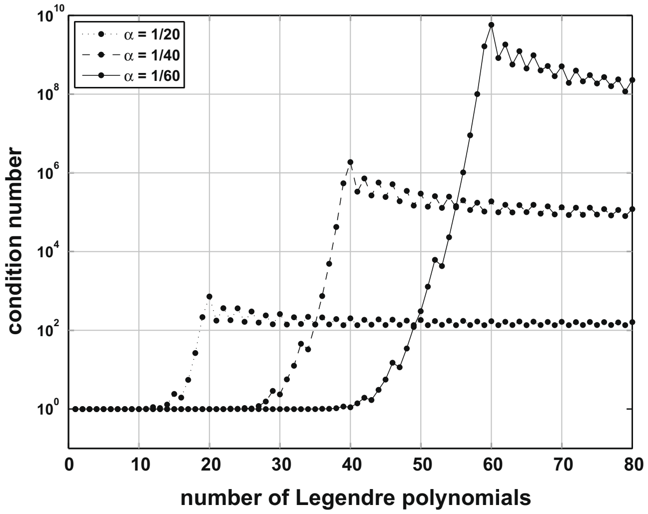

Fig. 2. The condition number of the matrix $A_{\lceil \alpha n^2 \rceil, n}$ for $\alpha = \frac{1}{20}, \frac{1}{40}, \frac{1}{60}$, and $n = 1, \ldots, 80$.

Fig. 1 shows the condition numbers of the matrices $A_{n,n}$ (IPRM) and $A_{n^2,n}$ (modified IPRM) for $n = 1, \ldots, 80$. It has been observed experimentally [11] that the condition number of the matrix $A_{n,n}$ grows exponentially with the dimension n. On the other hand, the condition number of $A_{n^2,n}$ remains bounded, with the maximum value of 1.21 attained at $n = 2$.

In order to understand the behavior of the matrices $A_{m,n}$ for a wider range of the parameters m and n, we have calculated the condition numbers of the matrices $A_{\lceil \alpha n^2 \rceil, n}$ as a function of n for three small values of α. Fig. 2 demonstrates that the condition numbers remain bounded as n approaches infinity, although Theorem 4.3 does not apply any more. The bound depends on the proportionality factor α.

Based on these results, we anticipate that the condition number of the matrix $A_{m,n}$ does not exceed $c_1 e^{c_2 n^2/m}$ for some absolute constants $c_1, c_2 > 0$. This behavior is consistent with the exponential growth of the condition number of $A_{n,n}$ (Fig. 1), the boundedness of the condition number of $A_{\lceil \alpha n^2 \rceil, n}$ for $\alpha \geqslant 1$ stated in Theorem 4.3, and with the simulations presented in Fig. 2.

6.2. Fourier reconstruction

In this subsection, we compare our algorithm with the most stable version of the IPRM, which uses the Gaussian elimination with truncation of all sufficiently small entries in the solution of the lower triangular stage, see [11] for details.

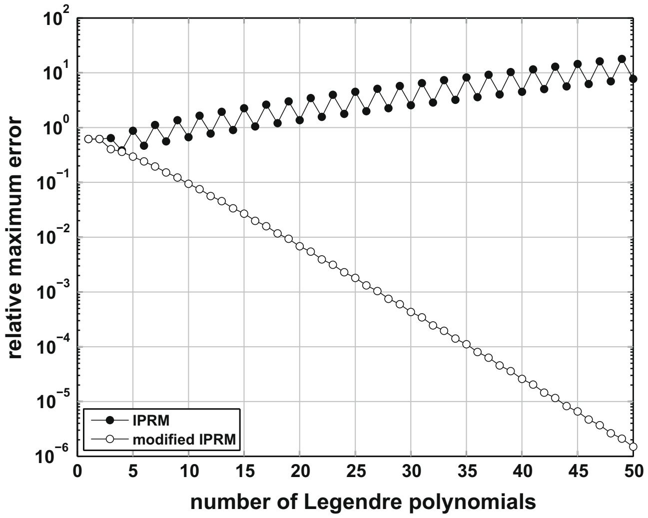

Fig. 3. The relative maximum errors of reconstruction with the IPRM and with the modified IPRM for the function $f(x) = \frac{1}{x - 0.3i}$.

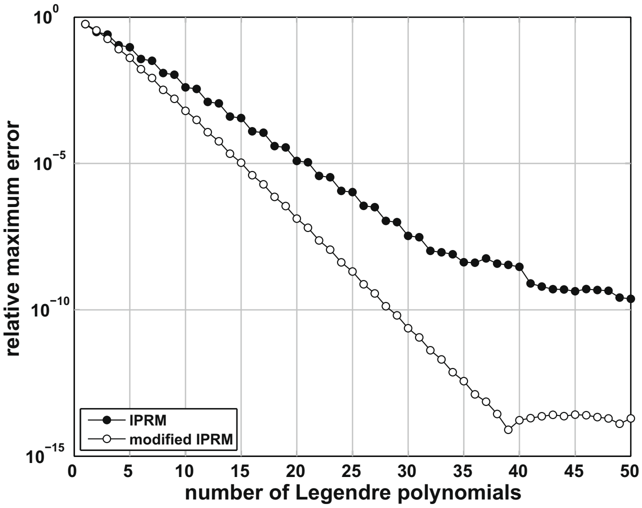

Fig. 4. The relative maximum errors of reconstruction with the IPRM and with the modified IPRM for the function $g(x) = \frac{1}{x-1.0\imath}$.

Fig. 3 shows the relative maximum errors $\|f - f_{m,n}\|_\infty / \|f\|_\infty$ for $m = n$ and $m = n^2$ for the function $f(x) = \frac{1}{x-0.3\imath}$ defined on the interval $[-1, 1]$. The modified IPRM converges exponentially in the number of Legendre coefficients in agreement with Theorem 5.1. On the other hand, the traditional IPRM appears to diverge. The reason for divergence can be discovered by looking at the analytic continuation of f, which has a singularity at $z = 0.3\imath$. Consequently, the Legendre series of f does not converge fast enough to mitigate the large condition numbers of the matrices $A_{n,n}$. The difference between the Fourier coefficients of f and those of its truncated Legendre series is amplified, and ultimately prevents convergence of IPRM.

Figs. 4 and 5 show, respectively, the relative maximum errors for the function $g(x) = \frac{1}{x-1.0\imath}$ and the function $h(x) = \exp(\sin(2.7x) + \cos x)$, both defined on the interval $[-1, 1]$. The latter function was used as a benchmark in [7]. In both cases, the errors of the modified IPRM steadily decrease to about 1e−14, where they level off.

6.3. Computing the pseudoinverse

In the modified IPRM algorithm, we apply the pseudoinverse $A^\dagger_{m,n}$ to the m lowest Fourier coefficients of f iteratively with the conjugate gradient method applied to the normal equations, specifically with the classical LSQR algorithm, see [18]. In order to experimentally test the convergence rate of this approach, we apply the LSQR iterations to the standard basis in $\mathbb{R}^{n^2}$, and in this way create the n^2 columns of the pseudoinverse matrix $A^\dagger_{n^2,n}$. Fig. 6 shows the relative errors in the operator norm of computing $A^\dagger_{n^2,n}$ with LSQR for $n = 5$ and $n = 50$. In experiments with other values of n, five iterations of LSQR are sufficient

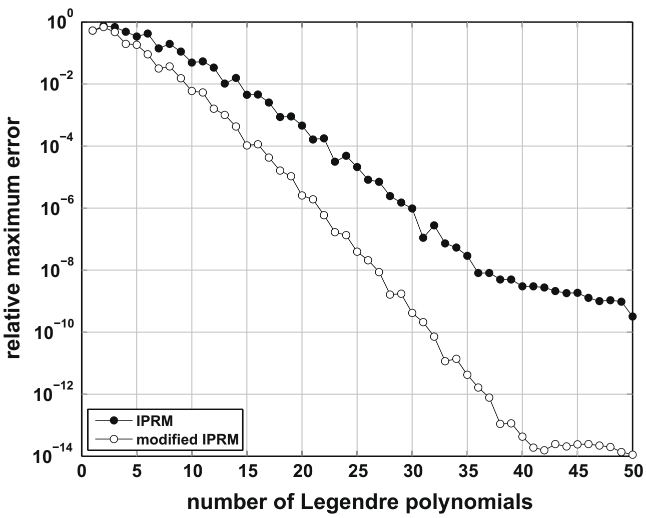

Fig. 5. The relative maximum errors of reconstruction with the IPRM and with the modified IPRM for the function $h(x) = \exp(\sin(2.7x) + \cos x)$.

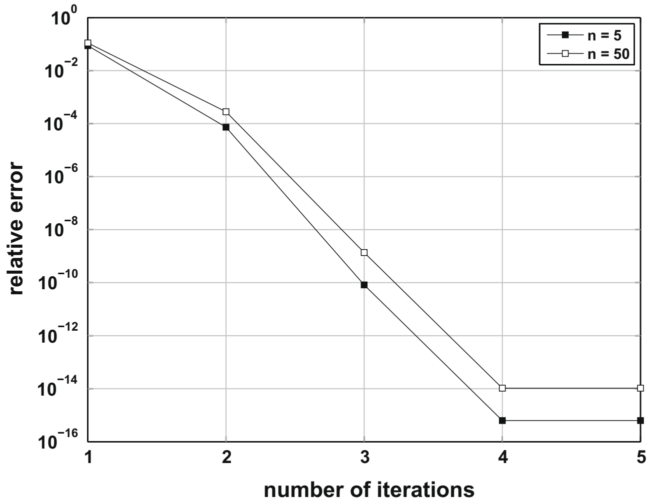

Fig. 6. The relative errors of computing $A^{\dagger}_{n^2,n}$ with LSQR for $n = 5$ and $n = 50$.

to apply the pseudoinverse matrix in double precision. The rapid convergence of LSQR is a result of well-conditioning of the matrices $A_{n^2,n}$, see Section 5.3.

6.4. Solution of the Poisson equation

Following an example in [14], we consider the Dirichlet problem for the one-dimensional Poisson equation

$$u_{xx} - u = -f(x). \tag{77}$$

We choose $f(x) = \frac{1}{x-a}$ on the interval $[-1, 1]$, where a is a given parameter. One particular solution u_0 to (77) is given by the Fourier series

$$u_0(x) = \sum_{k=-\infty}^{\infty} \frac{\hat{f}(k)}{1 + \pi^2 k^2} e^{\imath \pi k x}. \tag{78}$$

The Dirichlet boundary conditions $u(-1) = u(1) = 0$ are imposed by setting

$$u(x) = u_0(x) + Ae^{x} + Be^{-x}, \tag{79}$$

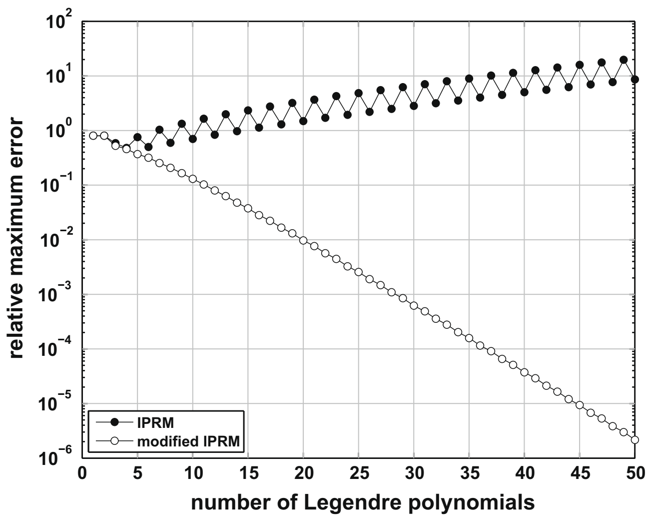

Fig. 7. The relative maximum errors of the solution of the Dirichlet problem for the equation $u_{xx} - u = -\frac{1}{x - 0.3\imath}$ with the IPRM and with the modified IPRM.

with appropriately chosen constants A and B. The function $f(x)$ is non-periodic, and its Fourier series converges rather slowly. More precisely, $\hat{f}(k) = \mathcal{O}(\frac{1}{|k|})$, and $\widehat{u}_0(k) = \mathcal{O}\left(\frac{1}{|k|^3}\right)$ as $|k| \to \infty$. However, one can reconstruct u_0 with the modified IPRM, and obtain a root-exponential convergence rate. Fig. 7 presents the relative errors of the solution u for $f(x) = \frac{1}{x-0.3i}$ with the IPRM and with the modified IPRM. We note that the rates of convergence and divergence in Fig. 7 are very close to those in Fig. 3.

7. Conclusions

In this article we introduce a modification of the IPRM for mitigation of the Gibbs phenomenon. The proposed algorithm achieves a tight control of the condition numbers by reconstructing only $\mathcal{O}(\sqrt{m})$ Legendre coefficients from given m Fourier coefficients. As a result, for analytic functions the convergence rate is root exponential on the whole interval. The algorithm requires $O(m^{1.5}\log\varepsilon)$ flops for convergence to the relative precision of ε.

Several pertinent questions remain open. There is experimental evidence that $n = \mathcal{O}(\sqrt{m})$ is required for uniform boundedness of the condition numbers of the matrices $A_{m,n}$. Specifically, we conjecture that if $n \approx m^p$ for some constant $p > \frac{1}{2}$, then the condition numbers of the matrices $A_{m,n}$ become arbitrarily large as m approaches infinity.

Additionally, we conjecture that the condition number of the matrix $A_{m,n}$ does not exceed $c_1 e^{c_2 n^2/m}$ for some absolute constants $c_1, c_2 > 0$.

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Appendix A. In this section we demonstrate a technical lemma used in Section 4.

Lemma 8.1. For every $u > 0$ and every $M \geqslant u + \frac{1}{2}$

$$\sum_{k \geqslant M} \frac{1}{k\sqrt{k^2 - u^2}} \leqslant \frac{1}{u} \arcsin \frac{u}{M - \frac{1}{2}}. \tag{80}$$

Proof. The function $f(x) = \frac{1}{x\sqrt{x^2-u^2}}$ is convex on the interval (u, ∞), so for every $k \geqslant u + \frac{1}{2}$ the following inequality holds

$$\frac{1}{k\sqrt{k^2 - u^2}} \leqslant \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \frac{dx}{x\sqrt{x^2 - u^2}}. \tag{81}$$

Summing inequality (81) over $k \geqslant M$, we obtain

$$\sum_{k \geqslant M} \frac{1}{k\sqrt{k^2 - u^2}} \leqslant \int_{M-\frac{1}{2}}^{\infty} \frac{dx}{x\sqrt{x^2 - u^2}} = \frac{1}{u} \arcsin \frac{u}{M - \frac{1}{2}}. \quad \square \tag{82}$$

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